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On the Laplacian flow and coflow of G_2-structures

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Abstract. We review some recent results on the study of the Laplacian flow and coflow of G_2-structures.

Keywords: G_2-structures, Laplacian flow, Laplacian coflow, Lie groups, warped products.
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§1. Introduction

In the 50's Berger [3] obtained the list of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds. In that list for the particular case of 7-dimensional manifolds appeared the exceptional holonomy Lie group G_2. A first tool in order to describe manifolds with holonomy G_2 is the concept of G_2-structure introduced by Bonan in [4]. A G_2-structure on a 7-dimensional manifold M can be characterized by the existence of a certain globally defined 3-form σ which is called the fundamental 3-form. The presence of such a structure on a manifold defines a metric g_σ on it, a volume form, and hence a Hodge star operator, namely $*$. Fernández and Gray in [12] gave a characterization for a manifold endowed with a G_2-structure to have holonomy restricted to the group G_2.

Theorem 1. *[12]. Let M be a manifold endowed with the G_2-structure σ. Denote by ∇^σ the Levi-Civita connection of the metric induced by the G_2-structure. Then, the following conditions are equivalent:*

- $Hol(\nabla^\sigma) \subseteq G_2$.
- $\nabla^\sigma \sigma = 0$.
- $d\sigma = d * \sigma = 0$.

The problem of obtaining manifolds with holonomy group G_2 was not a straightforward task and until the 80's the first examples were not described. In particular the first local example is due to Bryant [5], and later in a joint work with Salamon [6] obtained the first complete examples. These examples are obtained by considering 7-dimensional manifolds endowed with SO(3) or SO(4)-structures and a splitting of type 3+4. On those manifolds can be described a G_2-structure σ such that $d\sigma = 0$ and $d * \sigma = 0$. Concerning compact examples with holonomy G_2 the first ones were described by Joyce in [20] using the Kümmer construction for K3 surfaces. Later, Kovalev [22] and more recently Corti, Haskins, Nordstrom and Pacini have obtained new compact examples of manifolds with holonomy G_2 with the twisted connected sum construction and an extension of that technique respectively.

The torsion of a G_2-structure can be identified with the covariant derivative of the fundamental form σ and, as it is described in [12], it can be decomposed into four G_2 irreducible components, namely X_1, X_2, X_3 and X_4. Thus, a G_2-structure is said to be of type

$\mathcal{P}, \mathcal{X}_i, \mathcal{X}_i \oplus \mathcal{X}_j, \mathcal{X}_i \oplus \mathcal{X}_j \oplus \mathcal{X}_k$ or $\mathcal{X}$ if the covariant derivative $\nabla^\sigma \sigma$ lies in $\{0\}, X_i, X_i \oplus X_j, X_i \oplus X_j \oplus X_k$ or $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, respectively. Hence, there exist 16 different classes of G_2-structures.

Another technique that allows to obtain examples of manifolds with holonomy in the group G_2 is via the study of flows of G_2-structures. These flows consist on one-parameter families of G_2-structures with certain initial conditions and such that satisfy an appropriated evolution equation. If this evolution equation is chosen appropriately, a solution for that flow is such that the initial value for the G_2-structures, which can have torsion, evolves to a G_2-structure without torsion. In this note we summarize some known results concerning the study of flows of G_2-structures, concretely we focus our attention on the Laplacian flow and the Laplacian coflow of a G_2-structure.

§2. Preliminars

We start explainning the basics about SU(3) and G_2-structures which are helpful for a brief introduction to the topic.

2.1. G_2-structures

A G_2-structure on a 7-dimensional manifold M consists of a reduction of the structure group of its frame bundle to the Lie group G_2. The existence of such structure on a manifold M can also be characterized by the presence of a global non-degenerate 3-form σ which can be locally written as

$$\sigma = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{1}$$

where $\{e^1, \dots, e^7\}$ is a local basis of 1-forms on M which we call the adapted basis. As usual in the related literature the notation $e^{i_1 \cdots i_k}$ stands for the wedge product $e^{i_1} \wedge \cdots \wedge e^{i_k}$.

A manifold M endowed with a G_2-structure σ is called a G_2 manifold and the corresponding structure defines also a volume form vol_7 and a Riemannian metric g_σ satisfying

$$g_\sigma(X, Y) vol_7 = \frac{1}{6} \iota_X \sigma \wedge \iota_Y \sigma \wedge \sigma,$$

for every X, Y vector fields on M.

In order to describe the different classes of G_2-structures we consider first the G_2 type decomposition of the space of forms (see [5] for details). Let (M, σ) be a G_2 manifold, consider the action of the group G_2 on the space of differential p-forms on the manifold M, namely $\Omega^p(M)$. This action is irreducible on $\Omega^1(M)$ and $\Omega^6(M)$, but it is reducible for $\Omega^p(M)$ with $2 \le p \le 5$. The G_2 irreducible decompositions for $p = 2$ and 3 are

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M),$$

where those irreducible spaces can characterized by

$$\begin{aligned} \Omega^2_7(M) &= \{*_7(\alpha \wedge *_7\sigma) \mid \alpha \in \Omega^1(M)\}, \\ \Omega^2_{14}(M) &= \{\beta \in \Omega^2(M) \mid \beta \wedge \sigma = - *_7 \beta\} = \{\beta \in \Omega^2(M) \mid \beta \wedge *_7\sigma = 0\}, \end{aligned}$$

Class	Torsion forms	Condition	Structure
$\mathcal{P}$	$\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$	$d\sigma = d *_7 \sigma = 0$	Parallel
$\mathcal{X}_2$	$\tau_0 = \tau_1 = \tau_3 = 0$	$d\sigma = 0$	Closed
$\mathcal{X}_4$	$\tau_0 = \tau_2 = \tau_3 = 0$	$d\sigma = 3\tau_1 \wedge \sigma$, $d *_7 \sigma = 4\tau_1 \wedge *_7\sigma$	Locally Conformal Parallel
$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\tau_1 = \tau_2 = 0$	$d *_7 \sigma = 0$	Coclosed
$\mathcal{X}_2 \oplus \mathcal{X}_4$	$\tau_0 = \tau_3 = 0$	$d\sigma = 3\tau_1 \wedge \sigma$	Locally Conformal Closed

Table 1: Principal classes of G_2-structures

and

$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),$$

with

$$\begin{aligned}
\Omega^3_1(M) &= \{f\sigma \mid f \in C^\infty(M)\},\\
\Omega^3_7(M) &= \{*_7(\alpha \wedge \sigma) \mid \alpha \in \Omega^1(M)\},\\
\Omega^3_{27}(M) &= \{\gamma \in \Omega^3(M) \mid \gamma \wedge \sigma = 0,\ \gamma \wedge *_7\sigma = 0\},
\end{aligned}$$

where $\Omega^p_k(M)$ denotes a G_2 irreducible space of p-forms of dimension k at every point. Note that the description on the other degrees are obtained via the isomorphism described by the Hodge star operator, i.e. $*_7\, \Omega^p_k(M) \cong \Omega^{7-p}_k(M)$.

The G_2 type decomposition of forms on M allows to express the exterior derivative of σ and $*_7\sigma$ as follows

$$\begin{aligned}
d\sigma &= \tau_0 *_7\sigma + 3\,\tau_1 \wedge \sigma + *_7\,\tau_3,\\
d *_7\sigma &= 4\,\tau_1 \wedge *_7\sigma + \tau_2 \wedge \sigma,
\end{aligned} \tag{2}$$

where $\tau_0 \in C^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ are called the torsion forms of the G_2-structure.

Notice that all the information of the torsion of a G_2-structure is encoded on the covariant derivative of the fundamental form σ but also on the exterior derivatives of σ and $*\sigma$. Thus the different classes of G_2-structures can be described in terms of their behavior or equivalently, in view of (2), by the torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$. In Table 1 some Fernández-Gray classes of G_2-structures are given.

The presence of certain G_2-structures on a manifold give information concerning its geometrical properties. Manifolds endowed with a parallel G_2-structure have holonomy contained in G_2, manifolds with a closed G_2-structure have non-positive scalar curvature. However, the scalar curvature of a manifold endowed with a coclosed G_2-structure has no sign restrictions. Locally Conformal Parallel and Locally Conformal Closed G_2-structures are (locally) Parallel and Closed G_2-structures which can be described by a conformal change of the original G_2-structure.

2.2. SU(3)-structures

An SU(3)-structure on a 6-dimensional manifold N consists of a triple (g, J, Ψ) such that g is a Riemannian metric, J is an almost complex structure compatible with the metric, and Ψ is

Class	Condition	Structure
$\{0\}$	$d\omega = d\psi_+ = d\psi_- = 0$	Calabi-Yau
$\mathcal{W}_1^-$	$d\omega = 3\psi_+, d\psi_- = -2\omega^2$	Nearly Kähler
$\mathcal{W}_2^-$	$d\omega = d\psi_+ = 0$	Symplectic half-flat
$\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$	$d\omega^2 = d\psi_+ = 0$	Half-flat

Table 2: Principal classes of SU(3)-structures

a complex volume form satisfying

$$\frac{3}{4} i \Psi \wedge \overline{\Psi} = \omega^3,$$

where ω is the fundamental form associated to the almost Hermitian structure (g, J). Note that an SU(3)-structure on a 6-dimensional manifold N can be described by the pair (ω, ψ_+), where ψ_+ is the real part of the complex volume form Ψ. Indeed, for the imaginary part ψ_- of the form Ψ one has that $\psi_- = J\psi_+$, so ψ_- is determined by ψ_+ and the almost complex structure J (see [18]). We will denote by g_{ω,ψ_+} the Riemannian metric induced by the SU(3)-structure.

Note that SU(3) and G_2-structures are closely related, in particular the presence of an SU(3)-structure (ω, ψ_+), on a 6-dimensional manifold N induces a G_2-structure on the 7-dimensional manifold $N \times L$ with $L = \mathbb{R}$ or S^1 which can be defined by

$$\sigma = \omega \wedge ds + \psi_+,$$

being s the coordinate on L.

As it is described in [9] the torsion of an SU(3)-structure, namely T, is identified with the covariant derivatives of ω and J and lies in a space of the form

$$T \in \mathcal{W}_1^\pm \oplus \mathcal{W}_2^\pm \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where $\mathcal{W}_i$ are the irreducible components under the action of the group SU(3). Analogously than for the G_2 case, this torsion can also be given in terms of the derivatives of the forms ω, ψ_+ and ψ_-. Equivalently the torsion forms of an SU(3)-structure can be defined (see [2] for details), but we will not care about this description on this note.

There exist many different classes of SU(3)-structures but the most relevant in the construction of G_2-structures are given in Table 2.

Calabi-Yau manifolds have holonomy in the group SU(3). Concerning nearly Kähler SU(3)-structures, not many examples of manifolds endowed with such structure are known, see [8] for homogeneous examples or in [16] can be found complete inhomogeneous examples on S^6 and $S^3 \times S^3$. Other well-known SU(3)-structures are the half-flat ones. These structures were first considered in [19] (see also [9]) and can be evolved to a parallel G_2-structure. Symplectic half-flat structures have been considered for several authors (see, for example, [10] and [13]) in order to obtain closed G_2-structures.

§3. Laplacian flow and coflow

The first author considering flows of G_2-structures was Bryant in [5]. The objective of considering flows of G_2-structures was to obtain examples of G_2-structures without torsion as the result of certain evolution of other G_2-structures with torsion. Thus, Bryant considered the so-called Laplacian flow of a G_2-structure σ_0 which is given by

$$\begin{cases} \frac{d}{dt}\sigma(t) = \Delta_t\sigma(t), \\ \sigma(0) = \sigma_0, \\ d\sigma(t) = 0, \end{cases} \tag{3}$$

where Δ_t denotes the corresponding Hodge Laplacian operator. On compact manifolds short time existence and uniqueness of solution for the Laplacian flow of a closed G_2-structure has been proved by Bryant and Xu in [7]. Xu and Ye in [29] proved long time existence and convergence of solution of the Laplacian flow starting near a torsion-free G_2-structure. In the last years Lotay and Wei in the series of papers [25, 26, 27] have obtained important results concerning long time existence and convergence of solution of the Laplacian flow.

On the other hand, in [21] Karigiannis, McKay and Tsui introduced the Laplacian coflow. This latter flow can be considered as the analogue to the Laplacian flow in which the fundamental 3-form is claimed to be coclosed instead of closed. Thus, this flow is given by the equations

$$\begin{cases} \frac{d}{dt}\psi(t) = -\Delta_t\psi(t), \\ \psi(0) = \psi_0, \\ d\psi(t) = 0, \end{cases}$$

with $\psi(t) = *_t\sigma(t)$ and $*_t$ denoting the Hodge star operator. As far as the authors know, short time existence and uniqueness of solution for this latter flow is not known. In [17] Grigorian introduced a modified version of this flow called modified Laplacian coflow for which he proved short time existence and uniqueness of solution.

3.1. Solutions of the Laplacian flow and coflow on Lie groups

The first examples of long time existence of solution for the Laplacian flow of closed G_2-structures were described in [11]. Concretely those examples are nilpotent Lie groups endowed with a one parameter family of left-invariant closed G_2-structures.

Theorem 2. *[11]. Consider the simply connected Lie group with Lie algebra given by the structure equations*

$$de^5 = e^1 \wedge e^2, \quad de^6 = e^1 \wedge e^3, \text{ and } de^i = 0 \text{ for all } i = 1, 2, 3, 4, 7.$$

The family of closed G_2 forms $\sigma(t)$ on N given by

$$\sigma(t) = e^{147} + e^{267} + e^{357} + f(t)^3 e^{123} + e^{156} + e^{245} - e^{346}, \qquad t \in \left(-\frac{3}{10}, +\infty\right),$$

where $f(t)$ is the function

$$f(t) = \Big(\frac{10}{3}t + 1\Big)^{\frac{1}{5}}.$$

is the solution of the Laplacian flow (3) *with initial value*

$$\sigma_0 = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets, as t goes to infinity.

More examples of long time solutions can also be found in [11] or in [23, 24]. Analogously in [1] have been given explicit long time solutions for the Laplacian coflow and the modified Laplacian coflow. These examples consist of one-parameter families of left-invariant coclosed G_2-structures on the 7-dimensional Heisenberg Lie group H_7 which is given by the matrices of the form

$$a = \begin{pmatrix} 1 & x_1 & x_3 & x_5 & x_7 \\ & 1 & & & x_2 \\ & & 1 & & x_4 \\ & & & 1 & x_6 \\ & & & & 1 \end{pmatrix}$$

with $x_i \in \mathbb{R}$ for all $i = 1, \dots 7$. Then a global system of coordinates x_i for H_7 is defined by $x_i(a) = x_i$. A standard calculation shows that a basis for the left invariant 1-forms on H_7 can be described by

$$\begin{aligned} &e^1 = dx_1, \quad e^2 = dx_2, \quad e^3 = dx_3, \quad e^4 = dx_4, \\ &e^5 = dx_5, \quad e^6 = dx_6, \quad \text{and} \quad e^7 = dx_7 - x_1 dx_2 - x_3 dx_4 - x_5 dx_6. \end{aligned}$$

Thus, the corresponding Lie algebra, namely $\mathfrak{h}_7$ is given by the structure equations

$$de^7 = -e^1 \wedge e^2 - e^3 \wedge e^4 - e^5 \wedge e^6, \text{ and } de^i = 0 \text{ for all } i = 1, \dots, 6.$$

Theorem 3. *[1]. Consider H_7 the 7-dimensional Heisenberg Lie group. Then, the solution of the Laplacian coflow on H_7 with the initial coclosed* G_2 *form,*

$$\sigma_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

is given by

$$\sigma(t) = \frac{1}{f(t)}(e^{127} + e^{347} + e^{567}) + f(t)^3(e^{135} - e^{146} - e^{236} - e^{245}), \quad t \in \Big(-\infty, \frac{3}{5}\Big)$$

where $f(t)$ is the positive function

$$f(t) = \Big(1 - \frac{5}{3}t\Big)^{\frac{1}{10}}.$$

Recently the study of the Laplacian flow and coflow of G_2-structures on Lie groups has been extended to different classes of G_2-structures like Locally Conformal Parallel G_2-structures (LCP for short) or Locally Conformal Closed ones (LCC for short). In particular, in [28] the authors consider the Laplacian flow, resp. coflow, of a LCP G_2-structure which can be defined as:

$$\begin{cases} \dfrac{d}{dt}\sigma(t) = \Delta_t \sigma(t), \\ \sigma(0) = \sigma_0, \\ d\sigma(t) = 3\,\tau(t) \wedge \sigma(t), \\ d *_t \sigma(t) = 4\,\tau(t) \wedge *_t\sigma(t). \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\psi(t) = -\Delta_t \psi(t), \\ \psi(0) = \psi_0, \\ d\psi(t) = 4\,\tau(t) \wedge \psi(t), \\ d *_t \psi(t) = 3\,\tau(t) \wedge *_t\psi(t), \end{cases}$$

obtaining the following results:

Theorem 4. *[28]. Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel* G_2 *form,* σ_0*, admits a long time solution* $\sigma(t)$ *to the Laplacian flow, preserving the LCP condition along the flow, such that* $\sigma(0) = \sigma_0$*.*

Theorem 5. *[28]. Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel* G_2 *form admits a long time LCP solution to the Laplacian coflow.*

On the other hand the Laplacian flow of LCC G_2-structures can be described by

$$\begin{cases} \frac{d}{dt}\sigma(t) = \Delta_t\, \sigma(t), \\ d\,\sigma(t) = 3\tau(t) \wedge \sigma(t), \\ \sigma(0) = \sigma_0. \end{cases} \tag{4}$$

For this latter flow explicit examples of long time solutions are given in [14].

Theorem 6. *[14]. Consider the simply connected, solvable Lie group whose Lie algebra has structure equations*

$$de^1 = \frac{1}{2}e^1 \wedge e^7, \qquad de^2 = \frac{1}{2}e^2 \wedge e^7, \qquad de^3 = \frac{1}{2}e^3 \wedge e^7, \qquad de^4 = \frac{1}{2}e^4 \wedge e^7,$$
$$de^5 = e^1 \wedge e^4 + e^2 \wedge e^3 + e^5 \wedge e^7, \quad de^6 = e^1 \wedge e^3 - e^2 \wedge e^4 + e^6 \wedge e^7, \text{ and } de^7 = 0.$$

The family of locally conformal closed G_2*-structures* $\sigma(t)$ *given by*

$$\sigma(t) = (1-4t)^{3/4}\, e^{127} + (1-4t)^{3/4}\, e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \text{ where } t \in \left(-\infty, \tfrac{1}{4}\right)$$

is the solution for the Laplacian flow (4) *of the* G_2 *form*

$$\sigma_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

The Lee 1-form $\theta(t)$ *of* $\sigma(t)$ *is* $\theta(t) = -e^7$*. Moreover, the underlying metrics* $g(t)$ *of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets, as t goes to* $-\infty$*, and they blow-up as t goes to* $\frac{1}{4}$*.*

3.2. Solutions of the Laplacian flow and coflow on warped products

Solutions of the Laplacian flow and coflow have also been obtained using warped products. The warped product of two Riemannian manifolds (F, g_F) and (B, g_B) is denoted by $B \times_f F$ and consists on the product manifold $B \times F$ endowed with the metric $g = \pi_1^*(g_B) + f^2\pi_2^*(g_F)$ with f a non-vanishing real differentiable function on B and π_1, π_2 the projections of $B \times F$ onto B and F, respectively.

As it is described in [15] if we consider $(\omega, \psi_\pm)$ an SU(3)-structure over a 6-dimensional manifold M^6 the 3-form

$$\sigma = f\omega \wedge ds + \psi_+$$

defines a G_2-structure on $M^7 = M^6 \times L$ with $L = \mathbb{R}$ or S^1 where f is a non-vanishing function on L and s the coordinate in L. This G_2-structure is called warped G_2-structure since the induced metric, namely g_σ, is exactly $g_{\omega,\psi_+} + f^2 ds^2$. Considering warped G_2-structures Fino and Raffero in [15] obtained sufficient conditions on the SU(3)-structure and the warping function f that guarantee the existence of solution for the Laplacian flow of a closed G_2-structure.

Concerning the Laplacian coflow of a coclosed G_2-structure Karigiannis, MacKay and Tsui in [21] showed that using warped products solutions for this flow could be obtained from 6-dimensional manifolds endowed with Nearly Kähler or Calabi Yau structures.

Let us finish by noticing that the Nearly Kähler or Calabi Yau conditions are very restrictive and thus not many examples of these classes are known. On the contrary with the approach of Fino and Raffero in [15] solutions for the Laplacian flow of a closed G_2-structure can be obtained from less restrictive conditions on the SU(3)-structure (concretely symplectic half-flat condition). Thus the following question naturally arises:

Question: Is it possible to obtain solutions for the Laplacian coflow as warped products of 6-dimensional manifolds endowed with less restrictive SU(3)-structures, like half-flat ones?

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